Problem Set 7 (Due 3/5/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Use formal division and multiplication of power series to determine the first 4 nonzero terms of following series. It is not necessary to show the steps of the division/multiplication, but you should briefly explain which power series are being dividing/multiplied, and why it is justified.

(a) The Laurent series of $\frac{1}{e^z-1}$ on the annulus $0 < |z| < 2\pi$.

(b) The Taylor series of $e^z \cos z$ about 0.

(c) The Laurent series of $\csc z = \frac{1}{\sin z}$ on the annulus $0 < |z| < \pi$.

Solution. (a) Consider the function $g(z) = \sum_{n=0} \frac{z^n}{(n+1)!}$. Since $g(z)$ is analytic and nonzero on the disk $|z| < 2\pi$, we can compute the Taylor series for $\frac{1}{g(z)}$ on that disk using formal long division. We will get

$$\frac{1}{\sum_{n=0} \frac{z^n}{(n+1)!}} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots .$$

The function $e^z - 1$ is analytic on the annulus $0 < |z| < 2\pi$ so we can use it's MacLaurin series. We have

$$\begin{aligned} \frac{1}{e^z-1} &= \frac{1}{\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1} \\ &= \frac{1}{\sum_{n=1}^{\infty} \frac{z^n}{n!}} \\ &= \frac{1}{z} \cdot \frac{1}{\sum_{n=0} \frac{z^n}{(n+1)!}} \\ &= \frac{1}{z}\left(1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots\right) \\ &= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \cdots . \end{aligned}$$

(b) Both functions are entire and hence have MacLaurin series that converge everywhere. Using formal multiplication of power series, we obtain

$$\begin{aligned} e^z \cos z &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right) \\ &= 1 + z - \frac{z^3}{3} - \frac{z^4}{6} + \cdots . \end{aligned}$$

(c) Consider the function $g(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}$. Since $g(z)$ is nonzero and analytic on the disk $|z| < \pi$, the function $\frac{1}{g(z)}$ has a Taylor series on that disk and the coefficients can be computed via formal long division. We will get

$$\frac{1}{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}} = 1 + \frac{z^2}{6} + \frac{7}{360}z^4 + \frac{31}{15120}z^6 + \cdots .$$

Since $\sin z$ is entire, we can use the MacLaurin series to obtain

$$\begin{aligned}\csc z = \frac{1}{\sin z} &= \frac{1}{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}} \\ &= \frac{1}{z} \cdot \frac{1}{\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!}} \\ &= \frac{1}{z}\left(1 + \frac{z^2}{6} + \frac{7}{360}z^4 + \frac{31}{15120}z^6 + \cdots\right) \\ &= \frac{1}{z} + \frac{z}{6} + \frac{7}{360}z^3 + \frac{31}{15120}z^5 + \cdots .\end{aligned}$$

P2. Let C be the positively oriented unit circle. Compute the following integrals. Justify your computations.

(a) $\int_C \frac{1}{e^z - 1}\,dz$;

(b) $\int_C e^z \cos z\,dz$;

(c) $\int_C \csc z\,dz$.

Solution. (a) The only singularity of the function that lies interior to C is $z = 0$. According to the residue theorem and the computation from P1(a),

$$\int_C \frac{1}{e^z - 1}\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{e^z - 1} = 2\pi i.$$

(b) This function is entire, so by the Cauchy-Goursat theorem we can conclude that the integral is zero.

(c) The only singularity of the function that lies interior to C is $z = 0$. According to the residue theorem and the computation from P1(c),

$$\int_C \csc z\,dz = 2\pi i \operatorname*{Res}_{z=0} \csc z = 2\pi i.$$

P3. For each function $f(z)$, compute $\operatorname*{Res}_{z=0} f(z)$. Justify your computations.

(a) $f(z) = z\cos\left(\frac{1}{z}\right)$;

(b) $f(z) = \frac{\csc z}{z^4}$.

Solution. (a) We seek a Laurent series on the annulus $0 < |z| < \infty$. Since $\cos z$ is entire, we can make use of it's MacLaurin series. We have

$$\begin{aligned} z\cos\left(\frac{1}{z}\right) &= z\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n}(2n)!} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n-1}(2n)!}. \end{aligned}$$

Evidently, then, $\operatorname{Res}_{z=0} z\cos\left(\frac{1}{z}\right) = -\frac{1}{2}$.

(b) Following the computation from P1(c), the Laurent series for $\frac{\csc z}{z^4}$ on the annulus $0 < |z| < \pi$ is given by

$$\frac{\csc z}{z^4} = \frac{1}{z^5} + \frac{1}{6z^3} + \frac{7}{360z} + \cdots.$$

Hence, $\operatorname{Res}_{z=0} \frac{\csc z}{z^4} = \frac{7}{360}$.

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P4. For each function $f(z)$, compute $\int_C f(z)\,dz$ where C is the positively oriented circle $|z| = \frac{31}{10}$.[1] Justify your computations.

(a) $f(z) = z\cos\left(\frac{1}{z}\right)$;

(b) $f(z) = \frac{\csc z}{z^4}$;

(c) $f(z) = \frac{1}{2-z}$;

(d) $f(z) = \frac{z}{(z-1)(z-3)}$;

(e) $f(z) = \frac{z}{1+z^2}$.

Solution. (a) By the residue theorem and P3(a), we have

$$\int_C z\cos\left(\frac{1}{z}\right)\,dz = 2\pi i \operatorname*{Res}_{z=0} z\cos\left(\frac{1}{z}\right) = -\pi i.$$

(b) The function is analytic everywhere on and interior to C except at the isolated singularity $z = 0$. By the residue theorem and P3(b), we have

$$\int_C \csc z\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{\csc z}{z^4} = \frac{7}{180}\pi i.$$

(c) The function is analytic everywhere on and interior to C except at $z = 2$. We seek a Laurent series on the annulus $0 < |z-2| < \infty$. In fact, the function $\frac{1}{2-z} = -\frac{1}{z-2}$ already involved only powers of $z-2$ so it is the Laurent series on that annulus. By the residue theorem,

$$\int_C \frac{1}{2-z}\,dz = 2\pi i \operatorname*{Res}_{z=2} \frac{1}{2-z} = -2\pi i.$$

WARNING: it would be incorrect to conclude from the computation in PSet 6 P3(b) that $\operatorname{Res}_{z=2} \frac{1}{2-z} = -1$ since the Laurent series in that problem was on the annulus $2 < |z| < \infty$ which is centered at 0 rather than 2.

[1]Hint: several of these functions appeared in PSet 6.

(d) The function is analytic everywhere on and interior to C except at the two isolated singularities $z = 1, 3$. The corresponding residues are easily read from the partial fraction decomposition in the first line of PSet 6 P3(d):

$$\frac{z}{(z-1)(z-3)} = \frac{3}{2} \cdot \frac{1}{z-3} - \frac{1}{2} \cdot \frac{1}{z-1}.$$

For instance, since $\frac{1}{z-1}$ is analytic at $z = 3$, it has a Taylor series there which will contribute no negative powers of $z-3$ to the Laurent series. Hence, $\operatorname{Res}_{z=3} \frac{z}{(z-1)(z-3)} = \frac{3}{2}$. There is no need to compute the entire Laurent series to compute the residue. Anyways, by the residue theorem, we have

$$\begin{aligned}\int_C \frac{z}{(z-1)(z-3)}\, dz &= 2\pi i \left(\operatorname*{Res}_{z=1} \frac{z}{(z-1)(z-3)} + \operatorname*{Res}_{z=3} \frac{z}{(z-1)(z-3)} \right) \\ &= 2\pi \left(-\frac{1}{2} + \frac{3}{2} \right) \\ &= 2\pi i.\end{aligned}$$

(e) Again, the function is analytic everywhere on and interior to C except at the isolated singularities $z = i, -i$. The residues are easily read from the partial fraction decomposition in the first line of PSet 6 P3(c):

$$\frac{z}{1+z^2} = \frac{1}{2} \left(\frac{1}{z+i} + \frac{1}{z-i} \right).$$

According to the residue theorem, we obtain

$$\begin{aligned}\int_C \frac{z}{1+z^2}\, dz &= 2\pi i \left(\operatorname*{Res}_{z=i} \frac{z}{1+z^2} + \operatorname*{Res}_{z=-i} \frac{z}{1+z^2} \right) \\ &= 2\pi i \left(\frac{1}{2} + \frac{1}{2} \right) \\ &= 2\pi i.\end{aligned}$$

P5. Let C be the unit circle with positive orientation. Let $n \geq 0$ be a nonnegative integer.

(a) The function $f(z) = z^n e^{\frac{1}{z}}$ has an isolated singularity at $z_0 = 0$. Find the Laurent series expansion for f on the annulus $0 < |z| < \infty$.

(b) Use the theory of residues to compute the integral

$$\int_C z^n e^{\frac{1}{z}}\, dz.$$

(c) Prove that[2]

$$\int_C e^{z+\frac{1}{z}}\, dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

[2]Hint: write $e^{z+\frac{1}{z}} = e^{\frac{1}{z}} \sum_{n=0}^{\infty} \frac{z^n}{n!}$, then apply the theorem on integrating power series.

Solution. (a) Using the MacLaurin series for the exponential function, we obtain

$$\begin{aligned} z^n e^{\frac{1}{z}} &= z^n \sum_{k=0}^{\infty} \frac{1}{z^k k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} z^{n-k}. \end{aligned}$$

(b) From (a), we have

$$\operatorname*{Res}_{z=0} z^n e^{\frac{1}{z}} = \frac{1}{(n+1)!}.$$

The function $z^n e^{\frac{1}{z}}$ is analytic everywhere interior to and on C except at $z = 0$. By the residue theorem, we have

$$\int_C z^n e^{\frac{1}{z}}\, dz = 2\pi i \operatorname*{Res}_{z=0} z^n e^{\frac{1}{z}} = \frac{2\pi i}{(n+1)!}.$$

(c) The function $g(z) = e^{\frac{1}{z}}$ is continuous on C and C lies interior to the disk of convergence of the series . Hence, using the theorem on integrating power series, we have

$$\begin{aligned} \int_C e^{z+\frac{1}{z}}\, dz &= \int_C e^{\frac{1}{z}} \sum_{n=0} \frac{z^n}{n!}\, dz \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n e^{\frac{1}{z}}\, dz \\ &= 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}. \end{aligned}$$

P6. Consider the function

$$f(z) = \frac{2z^3+1}{(z^2-1)(z^2+1)}.$$

Show that $f(z)$ has no antiderivative on the domain $D = \{z \in \mathbb{C} : |z| > 1\}$.[3]

Proof. The function $f(z)$ is continuous on D so it suffices, according to the Fundamental Theorem of Contour Integrals, to show that $\int_C f(z)\, dz \neq 0$ where C is the positively oriented circle of radius 2 centered at 0. Since C contains all the singularities of f in its interior, we can compute this integral with a single residue using the theorem on residues at ∞. According to that theorem, we have

$$\int_C f(z)\, dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2}\left(\frac{1}{z}\right) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z}\left(\frac{2+z^3}{1-z^4}\right).$$

To compute this residue, we seek a Laurent series centered at 0. Notice that the function $\frac{2+z^3}{1-z^4}$ is analytic at 0, so it has a MacLaurin series there whose constant term is equal to 2:

$$\frac{2+z^3}{1-z^4} = (2+z^3)(1+z^4+z^8+\cdots) = 2 + a_1 z + a_2 z^2 + \cdots.$$

[3]Hint: consider a circle C that lies in D and contains all the singularities of f in it's interior. Compute $\int_C f(z)\, dz$ by computing the residue at ∞. Use the theorem that says $\operatorname{Res}_{z=\infty} f(z) = \operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$.

Since this series has no negative powers of z, the only contribution to the residue is the constant term - the other coefficients are immaterial. Evidently then, after multiplying through by $\frac{1}{z}$, we see that

$$\operatorname*{Res}_{z=0} \frac{1}{z}\left(\frac{2+z^3}{1-z^4}\right) = 2$$

and

$$\int_C f(z)\,dz = 4\pi i \neq 0.$$

We conclude that $f(z)$ has no antiderivative on this domain.

P7. (Optional.)[4]

(a) Suppose that f is analytic everywhere on $\mathbb{C}$, except at a finite number of (isolated) singular points $z_1, \ldots, z_n$. Show that

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

(b) Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0$$

be a polynomial of degree $n \geq 2$. Let C be a positively oriented simple closed contour whose interior contains all the zeros of $p(z)$. Show that[5]

$$\int_C \frac{1}{p(z)}\,dz = 0.$$

(c) Using (a) and (b), compute the integral

$$\int_C \frac{1}{(z^{2021}+1)(z-3)}\,dz$$

where C is the positively oriented circle $|z| = 2$.

Solution. (a) To prove this, let C be a positively oriented circle centered at 0 with radius $R > \max_{i=1}^{n} |z_i|$ so that the singularities $z_1, \ldots, z_n$ lie interior to C. Then according to the residue theorem and the definition of the residue at ∞, we see that

$$\begin{aligned}\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) &= \int_C f(z)\,dz + \int_{-C} f(z)\,dz \\ &= \int_C f(z)\,dz - \int_C f(z)\,dz \\ &= 0.\end{aligned}$$

(b) Since C contains all of the zeros of $p(z)$ in it's interior, we can compute the integral using the theorem on residues at ∞. We have

$$\int_C \frac{1}{p(z)}\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} \cdot \frac{1}{p\left(\frac{1}{z}\right)}.$$

[4] Worth 4 points of extra credit, added to you score on this assignment.
[5] Hint: show that $\operatorname{Res}_{z=\infty} \frac{1}{p(z)} = 0$.

Then we have

$$\begin{aligned}\frac{1}{z^2}\cdot\frac{1}{p\left(\frac{1}{z}\right)} &= \frac{1}{z^2}\cdot\frac{1}{a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots+\frac{a_{n-1}}{z^{n-1}}+\frac{a_n}{z^n}} \\ &= \frac{z^{n-2}}{a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_{n-1}z+a_n}.\end{aligned}$$

Since $n \geq 2$ and $a_n \neq 0$, it is clear that this function is analytic at $z = 0$. Hence, the residue is zero which completes the proof of the claim.

(c) Let $z_1, z_2, \ldots, z_{2021}$ denote the distinct solutions to the equation $z^{2021} = -1$. Then each z_i is an isolated singularity that lies interior to C. Additionally, there is a single isolated singularity at $z = 3$ that lies exterior to C. Using the residue theorem, together with parts (a) and (b), we have

$$\begin{aligned}\int_C \frac{1}{(z^{2021}+1)(z-3)}\,dz &= 2\pi i\sum_{i=1}^{2021}\operatorname*{Res}_{z=z_k}\frac{1}{(z^{2021}+1)(z-3)} && \text{(Residue theorem)} \\ &= -2\pi i\left(\operatorname*{Res}_{z=3}\frac{1}{(z^{2021}+1)(z-3)}+\operatorname*{Res}_{z=\infty}\frac{1}{(z^{2021}+1)(z-3)}\right) && \text{(by part (a))} \\ &= -2\pi i\operatorname*{Res}_{z=3}\frac{1}{(z^{2021}+1)(z-3)} && \text{(by part (b))} \\ &= -\frac{2\pi i}{3^{2021}+1}.\end{aligned}$$

This last residue was computed as follows: write $p(z) = \frac{1}{z^{2021}+1}$ and $q(z) = z-3$ so that $\frac{1}{(z^{2021}+1)(z-3)} = \frac{p(z)}{q(z)}$. Clearly, both $p(z)$ and $q(z)$ are analytic at $z = 3$. Moreover,

$$p(3) = \frac{1}{3^{2021}+1} \neq 0, \quad q(3) = 0, \quad \text{and} \quad q'(3) = 1 \neq 0.$$

According to a theorem from class, this proves that $z = 3$ is a simple pole and residue is given by

$$\operatorname*{Res}_{z=3}\frac{1}{(z^{2021}+1)(z-3)} = \frac{p(3)}{q'(3)} = \frac{1}{3^{2021}+1}.$$

This completes the solution.